PARAMETRIC INSTABILITY OF TRANSVERSE VIBRATIONS OF A FILAMENT WHOSE PARAMETERS VARY ACCORDING TO A TRAVELING WAVE LAW

A. I. Vesnitskii and A. F. Lyakhov

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In connection with the continuous growth of requirements on the fast-response of machines, instruments, and apparatus in solving problems on raising the reliability and longevity of their operation it turns out to be more often essentially necessary to take account of the wave nature of dynamic processes in elastic elements and particularly the possibility of the origination of a parametric instability. Up to now distributed systems whose parameter (P) vary uniformly in time in the whole space [1, 2] ($P = P_1(t)P_2(\mathbf{r})$, where t is time and $\mathbf{r}$ is a radius-vector) have been investigated in sufficient detail. The class of distributed systems whose parameters vary inhomogeneously (e.g., system with moving boundaries, with boundaries of varying properties, with traveling, standing waves of parameters, etc.), has been studied quite little although it is very much broader than the former. Up to now a complete set of phenomena and effects they specify, which are possible in such systems, has remained unclarified.

One of the characteristic features of systems with inhomogeneously varying parameters is the possibility of exciting impulsive oscillations with a broad frequency spectrum therein [3]. In this connection, featues of the appearance of a parametric instability expressed as the transformation of initial perturbation into essentially nonharmonic waves having the shape of impulses are investigated in this paper in an example of the simplest one-dimensional system of filament (string) type whose parameters vary according to a traveling wave law. Such investigations are not only of theoretical but also of practical interest since the effect of impulsive vibration formation can be one of the reasons for dynamic buckling of transmission branches with flexible constraints [4], as well as the disturbance of technological spinning and weaving processes.

1. A parametric instability of a vibrational process in a linear system is expressed as a physical phenomenon by the unbounded growth of vibrations energy E with the lapse of time. Starting from this, we define the stability and instability of a linear parametric system appropriately.

Definition 1. A distributed linear bounded system with parameters varying in time and space will be stable if for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that if the total vibration energy of the system $E[0, x, y, z, u_n, u_{nt}, u_{nx}, u_{ny}, u_{nz}, l_i(0)] < \delta$ at the initial instant $t = 0$, then for any $t > 0$, $E[t, x, y, z, u_n, u_{nt}, u_{nx}, u_{ny}, u_{nz}, l_i(t)] < \varepsilon$, where $u_n(x, y, z)$ are generalized coordinates describing the given distributed system, x, y, z are the space coordinates, and $l_i(t)$ are parameters characterizing the system dimensions.

Definition 2. A distributed linear bounded system with parameters varying in time and space will be unstable if for any $\varepsilon > 0$ there exists a $t_0(\varepsilon)$ such that if for $t = 0$ the total vibration energy $E[0, x, y, z, u_n, u_{nt}, u_{nx}, u_{ny}, u_{nz}, l_i(0)] \neq 0$, then for $t > t_0(\varepsilon)$, $E[t, x, y, z, u_n, u_{nt}, u_{nx}, u_{ny}, u_{nz}, l_i(t)] > \varepsilon$.

Let us consider the problem of transverse vibrations of a flexible filament with distributed parameters varying in time and space

$$(\rho u_t)_t = (T u_x)_x; \tag{1.1}$$

$$u(0, t) = u(l, t) = 0; \tag{1.2}$$

$$u(x, 0) = \varphi(x), \ u_t(x, 0) = \psi(x). \tag{1.3}$$

Here u(x, t) is the transverse displacement; $\rho(x, t)$, linear density; T(x, t), filament tension. Multiplying (1.1) by u_t and combining with the identity $Tu_x(u_{xt} - u_{tx}) = 0$, we obtain the Umov–Poynting theory after appropriate grouping and taking account of the nonstationarity of the parameters*

*The Umov–Poynting theorem for systems with nonstationary parameters has been formulated earlier only in application to electrodynamic problems [5].

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$$\frac{\partial S}{\partial x} = -\frac{\partial W}{\partial t} = -\rho_t \frac{u_t^2}{2} + T_t \frac{u_x^2}{2}, \tag{1.4}$$

where $W = (\rho u_t^2 + Tu_x^2)/2$ is the energy density, $S = -Tu_x u_t$ is the wave energy flux (Poynting vector).

Taking into account that the energy flux through the system boundary is zero, in integral form theorem (1.4) has the form

$$\frac{\partial}{\partial t} E = \int_0^l \left(T_t \frac{u_x^2}{2} - \rho_t \frac{u_t^2}{2} \right) dx,$$

where $E = \int_0^l W dx$ is the total mechanical energy of the filament.

By using (1.4), a number of known results can be obtained and some new appearances of parametric instability in distributed systems can be clarified.

By Definition 2, the parametric instability of the filament will be observed if the work of the external forces transmitted to the system as the parameters change is positive. It follows from (1.4) that this is possible if the filament density diminishes ($\rho_t < 0$) and the filament tension increases ($T_t > 0$). In real systems the values of the tension and density are always bounded; hence, it is impossible to reach parametric instability of the vibrations because of a monotonic change in the parameters.

Let us examine the case when the filament parameters vary periodically over its whole length. Assuming the filament inextensible ($\rho_t = 0$) for simplicity and $T = T_0(1 - m \sin 2\Omega t)$ (m, T_0, and Ω are constants, where $m < 1$), we see from (1.4) that the standing waves in the filament will increase unlimitedly if their frequency is $\omega = \Omega$ (Fig. 1, where the energy inserted into the system by change in tension is shaded). In the case under consideration, an increase in the vibration energy is evidently possible only because of the growth in the vibration amplitude just as holds in lumped systems [6].

We agree to call the parametric instability that is accompanied by the unlimited growth of the vibration amplitude without a transformation in shape a parametric instability of the first kind.

Parametric instability of the system is possible even in the case when the parameters do not vary simultaneously over the whole length. Let us examine the propagation of perturbations in an infinite filament that are bounded in space, and whose parameters vary according to the traveling wave law [e.g., ρ = const, $T = T_0(1 + m \sin \Omega(t - x/a))$]. If the velocity of transverse wave propagation $c(x, t) = (T/\rho)^{1/2}$ is close to the propagation velocity of a wave of parameter a, then the initial perturbation can be arbitrarily long in a domain where $T_t > 0$ (Fig. 2). In conformity with (1.4), the initial perturbation energy will here grow without limit as it propagates, while the profile will be continuously transformed into a pulse (Fig. 2, t_1, t_2, t_3, t_4 are different times in which the perturbation is considered) because of the inhomogeneity of the gain coefficient ($\sim T_t$) and the propagation velocity $c(x, t)$.

We agree to call the parametric instability that is accompanied by continuous compression of the propagating wave profile a parametric instability of the second kind.

2. Let us consider the continuous amplification of propagating waves in an infinite filament accompanied by unbounded twisting of the profile. To this end we write (1.1) in the form of the system

$$V_x = \rho u_t, \quad V_t = T u_x.$$

Hence, by going over to the Riemann invariants $r_1 = u - V/(T\rho)^{1/2}$, $r_2 = u + V/(T\rho)^{1/2}$, we obtain

$$\begin{aligned} \frac{\partial r_1}{\partial t} + \sqrt{\frac{T}{\rho}} \frac{\partial r_1}{\partial x} &= \frac{r_1 - r_2}{2} \sqrt{T\rho} \left(\frac{\partial}{\partial t} \frac{1}{\sqrt{T\rho}} + \sqrt{\frac{T}{\rho}} \frac{\partial}{\partial x} \frac{1}{\sqrt{T\rho}} \right), \\ \frac{\partial r_2}{\partial t} - \sqrt{\frac{T}{\rho}} \frac{\partial r_2}{\partial x} &= \frac{r_2 - r_1}{2} \sqrt{T\rho} \left(\frac{\partial}{\partial t} \frac{1}{\sqrt{T\rho}} - \sqrt{\frac{T}{\rho}} \frac{\partial}{\partial x} \frac{1}{\sqrt{T\rho}} \right). \end{aligned} \tag{2.1}$$

If $T\rho$ = const (i.e., the wave resistance is constant), then $r_1 = f_1(\xi_1)$, $r_2 = f_2(\xi_2)$, where $\xi_1(x, t)$, $\xi_2(x, t)$ are the characteristics of the system (2.1).

Upon the formation of infinitely steep profiles of waves being propagated along x in the positive or negative directions, the derivatives of r_1 or r_2, respectively, with respect to the space coordinate

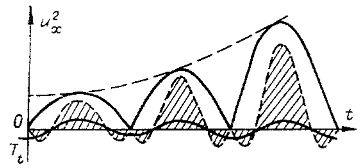

Fig. 1

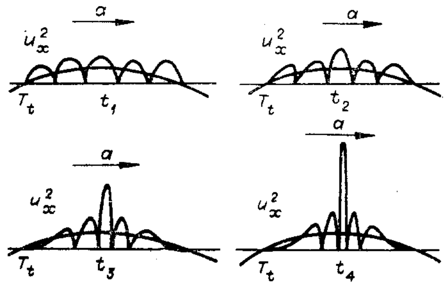

Fig. 2

$$\frac{\partial r_1}{\partial x} = \frac{df_1}{d\xi_1}\frac{\partial \xi_1}{\partial x}, \quad \frac{\partial r_2}{\partial x} = \frac{df_2}{d\xi_2}\frac{\partial \xi_2}{\partial x},$$

should tend to infinity as $t \to \infty$.

Assuming the initial conditions (1.3) from which the functions f_1 and f_2 are determined to be differentiable and bounded together with their derivatives, we obtain the condition of continuous twisting of the profile of one of the waves:

$$\partial \xi_i / \partial x_{t\to\infty} \to \infty (i = 1, 2), \tag{2.2}$$

which will also be the condition for the continuous coming together of curves of the families of characteristics in the x, t plane as $t \to \infty$. If the system parameters vary according to the traveling wave law $T = T(t - x/a)$, $\rho = \rho(t - x/a)$, then the characteristics ξ_1 and ξ_2 can be written in the explicit form

$$\xi_1 = x - \int_0^\eta \left(\frac{1}{c(\eta)} - \frac{1}{a}\right)^{-1} d\eta, \quad \xi_2 = x + \int_0^\eta \left(\frac{1}{c(\eta)} + \frac{1}{a}\right)^{-1} d\eta, \tag{2.3}$$

where $\eta = t - x/a$. We hence find

$$\partial x / \partial \xi_1 = (1 - c(\eta)/a), \quad \partial x / \partial \xi_2 = (1 + c(\eta)/a). \tag{2.4}$$

Since the wave propagation velocity will, as a characteristic of the medium, always be possible $[c(\eta) > 0]$, it follows from (2.2) and (2.4) that only the first family characteristics (2.3) can come unlimitedly close together.

As an illustration, let us consider the case when the parameters vary so that

$$c(\eta) = c_0(1 + m \cos \Omega\eta)^{-1}, \quad m < 1. \tag{2.5}$$

Depending on how close are the transverse wave propagation velocities and the wave parameters, two cases can be separated: 1) $|1 - c_0/a| < m$ when these waves are "synchronized," 2) $|1 - c_0/a| > m$ when they are "asynchronous." For the "synchronized" waves we have from (2.3)

$$\xi_1 = x - c_0\Omega^{-1}[m^2 - (1-\beta)^2]^{-\frac{1}{2}} \ln \left| \frac{\operatorname{tg}\frac{\Omega\eta}{2} + \gamma_0}{\operatorname{tg}\frac{\Omega\eta}{2} - \gamma_0} \right|, \tag{2.6}$$

and for the "asynchronous" waves

$$\xi_1 = x - 2c_0\Omega^{-1}[(1-\beta)^2 - m^2]^{-1/2} \operatorname{arctg}\left(\sqrt{\frac{1-\beta-m}{1-\beta+m}} \operatorname{tg}\frac{\Omega\eta}{2}\right),$$

where $\beta = c_0/a$; $\gamma_0 = [(m + 1 - \beta)/(m - 1 + \beta)]^{1/2}$. The second family of characteristics is identical for both cases

$$\xi_2 = x + 2c_0\Omega^{-1}[(1+\beta)^2 - m^2]^{-1/2} \operatorname{arctg}\left(\sqrt{\frac{1+\beta-m}{1+\beta+m}} \operatorname{tg}\frac{\Omega\eta}{2}\right).$$

It is hence seen that the condition for the characteristics (2.2) to come together is satisfied only in the case of the "synchronized" waves and only for the first family (2.6).

Twisting of the profile of the propagating waves is accompanied by growth in the "instantaneous frequencies" of these latter [7], i.e., by energy transmission upward in the spectrum. In order to see this, we consider the problem of the vibrations of a semiinfinite filament on whose boundary is a source $u(0, t) = \sin(\omega_0 t + \theta_0)$. Then, following the method of characteristics for waves being propagated in the +x direction, we can write $u(0, t) = \sin[\omega_0 t(\bar{\xi}_1) + \theta_0]$, where $\bar{\xi}_1 = \xi_1(0, t)$. Expressing $t = t(\xi_1)$ explicitly from the equation of the characteristics (2.6) and replacing $\bar{\xi}_1$ by ξ_1, we obtain

$$u(x, t) = \sin(\omega_0 t(\xi_1) + \theta_0),$$

$$t(\xi_1) = 2\Omega^{-1} \operatorname{arctg}\left[-\gamma_0 \frac{\operatorname{tg}\frac{\Omega\eta}{2}\left(1+e^{-\Omega\sqrt{m^2-(1-\beta)^2}x/c_0}\right) - \gamma_0\left(1-e^{-\Omega\sqrt{m^2-(1-\beta)^2}x/c_0}\right)}{\operatorname{tg}\frac{\Omega\eta}{2}\left(1-e^{-\Omega\sqrt{m^2-(1-\beta)^2}x/c_0}\right) - \gamma_0\left(1+e^{-\Omega\sqrt{m^2-(1-\beta)^2}x/c_0}\right)}\right].$$

Defining the "instantaneous frequency" as the partial derivative of the phase $\theta = \omega_0 t(\xi_1) + \theta_0$ with respect to time, we find at the distance $x_0 = \text{const}$

$$\omega = \left(\frac{\partial\theta}{\partial t}\right)_{x_0} = \omega_0 \frac{\partial t(\xi_1)}{\partial \xi_1}\frac{\partial \xi_1}{\partial t}\bigg|_{x_0}.$$

Taking into account that

$$\frac{\partial t(\xi_1)}{\partial \xi_1} = -\left(\frac{1}{c(t(\xi_1))} - \frac{1}{a}\right), \quad \frac{\partial \xi_1}{\partial t} = -\left(\frac{1}{c(\eta)} - \frac{1}{a}\right)^{-1},$$

we obtain

$$\omega = \omega_0 \frac{\frac{1}{c(t(\xi_1))} - \frac{1}{a}}{\frac{1}{c(\eta)} - \frac{1}{a}}\Bigg|_{x_0} = \omega_0 \frac{1 - a/c(t(\xi_1))}{1 - a/c(\eta)}\bigg|_{x_0}. \tag{2.7}$$

Expression (2.7) evidently describes the distributed Doppler effect for wave passage through a moving inhomogeneity of the medium parameters [8]. Substituting the value of $c[t(\xi_1)]$ and $c(\eta)$ in (2.7) we obtain

$$\omega = \omega_0 \frac{1 - \beta + m\cos\Omega t(\xi_1)}{1 - \beta + m\cos\Omega\eta}\bigg|_{x_0}.$$

It can be shown that when $\cos\Omega\eta \to (1-\beta)/m$

$$\omega \to \omega_0 \exp\left(\pm\sqrt{m^2 - (1-\beta)^2}\,\Omega x/c_0\right),$$

i.e., as the "instantaneous frequency" is propagated along x, the frequency corresponding to the section of increasing energy ($T_t > 0$) grows without limit. The energy density of the wave varies according to the law

$$W = \rho\omega_0^2\left(\frac{1 - a/c(t(\xi_1))}{1 - a/c(\eta)}\right)^2 \cos^2(\omega_0 t(\xi_1) + \theta_0)\big|_{x_0},$$

The following invariant holds for W

$$\overline{W}/\omega^2 = I,$$

where $\overline{W}$ is the energy density averaged over the period. The invariant I agrees in form with the invariant obtained in [9] for a wave interacting with a moving boundary. Setting $\omega_0 = \Omega$, $\beta = 1$, $\theta_0 = 0$, we have from (2.7)

$$u(x, t) = \frac{\operatorname{th}\Omega m x/c_0 - \sin\Omega(t - x/a)}{1 - \operatorname{th}\Omega x/c_0 \sin\Omega(t - x/a)}.$$

It follows from the expression obtained that as the wave is propagated in the filament, its profile is transformed continuously in a sequence of traveling pulses. The dependence of u(x, t) and $W \sim u_x^2$ on the coordinate x at a fixed time t is shown in Fig. 3.

3. Examination of processes of parametric instability of the second kind in systems of bounded length is of greatest practical interest. Setting $T\rho = \text{const}$, we write the solution of the problem (1.1)-(1.3) in the form

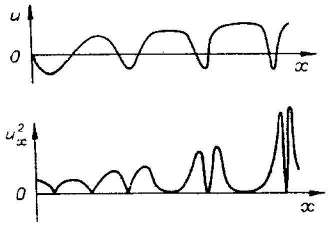

Fig. 3

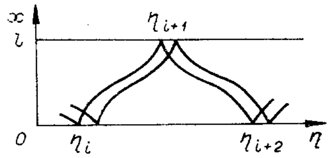

Fig. 4

$$u(x, t) = f_1(\xi_1) + f_2(\xi_2),$$

where $\xi_1(x, t)$, $\xi_2(x, t)$ are the characteristic equations of (1.1), and f_1, f_2 are determined from conditions (1.2) and (1.3).

As is shown above, wave transformation into a pulse train is accompanied by the characteristic coming together continuously on the x, t plane. Let us consider how the distance between two characteristics changes because of one reflection from the boundaries (Fig. 4). Using (2.3) we find

$$\left(\frac{1}{a}-\frac{1}{c(\eta_i)}\right)^{-1}d\eta_i=\left(\frac{1}{a}-\frac{1}{c(\eta_{i+1})}\right)^{-1}d\eta_{i+1},\quad \left(\frac{1}{a}+\frac{1}{c(\eta_{i+1})}\right)^{-1}d\eta_{i+1}=\left(\frac{1}{a}+\frac{1}{c(\eta_{i+2})}\right)^{-1}d\eta_{i+2}.$$

Hence $d\eta_{i+2}/d\eta_i = D_{i+1}^{+}D_i^{-}$,

$$D_{i+1}^{+}=\left(\frac{1}{a}+\frac{1}{c(\eta_{i+2})}\right)\left(\frac{1}{a}+\frac{1}{c(\eta_{i+1})}\right)^{-1},\quad D_i^{-}=\left(\frac{1}{a}-\frac{1}{c(\eta_{i+1})}\right)\left(\frac{1}{a}-\frac{1}{c(\eta_i)}\right)^{-1}.$$

$$\frac{d\eta_{2n}}{d\eta_0}=\prod_{i=0}^{2(n-1)}D_{i+1}^{+}D_i^{-},$$

and, therefore, the condition of interaction of broken lines comprised of segments of characteristics has the form as $t \to \infty$

$$\left(\prod_{i=0}^{2(n-1)}D_{i+1}^{+}D_i^{-}\right)_{n\to\infty}\to 0. \tag{3.1}$$

A change in the energy density occurs because of compression of the profile of the waves being propagated in the filament and of the work of external forces changing the system parameters. The vibration energy changes here in proportion to the frequency, i.e., the total number of quanta remains unchanged (E/ω = const). The criterion (3.1) is therefore a criterion of instability of the second kind.

In form it is analogous to the criterion that is obtained for a system with moving boundaries [10-12]. Comparing (3.1) and (2.7), as well as with the expression of the Doppler effect in passing, it can be seen that the first term under the product symbol in (3.1) characterizes the change in wave frequency during passage of a moving inhomogeneity in the medium parameters in the $-x$ direction, and the second factor is for the $+x$ direction.

The investigation of (3.1) in the general case is fraught with mathematical difficulties but if the broken line comprised of segments of characteristics and passing through the point $(0, \eta_i)$, (l, η_{i+1}), $(0, \eta_{i+2})$ is periodic ($\eta_{i+2} = \eta_i + \tau$, τ is the period of the broken line), then the criterion (3.1) will be satisfied if

$$\left|\frac{\left(\frac{1}{a}+\frac{1}{c(\eta_i)}\right)\left(\frac{1}{a}-\frac{1}{c(\eta_{i+1})}\right)}{\left(\frac{1}{a}+\frac{1}{c(\eta_{i+1})}\right)\left(\frac{1}{a}-\frac{1}{c(\eta_i)}\right)}\right|<1. \tag{3.2}$$

Seeking the conditions for the appearance of a parametric instability of the second kind in such a formulation dissociates into two stages: 1) finding the condition for the existence of a periodic broken line, and 2) finding the condition for the broken lines to come together.

1. By using (2.3), we write the system of equations governing the periodic broken line (see Fig. 4):

$$\xi_1 = x_0 - \int_0^{\eta_0}\left(\frac{1}{c(\eta)}-\frac{1}{a}\right)^{-1}d\eta,\quad \xi_1 = x_1 - \int_0^{\eta_1}\left(\frac{1}{c(\eta)}-\frac{1}{a}\right)^{-1}d\eta,$$

$$\xi_2 = x_1 + \int_0^{\eta_1} \left(\frac{1}{c(\eta)} + \frac{1}{a}\right)^{-1} d\eta, \; \xi_2 = x_2 + \int_0^{\eta_2} \left(\frac{1}{c(\eta)} + \frac{1}{a}\right)^{-1} d\eta,$$

$$x_2 = x_0 = 0, \; x_1 = l, \; \eta_2 = \eta_0 + \tau.$$

Hence there follows

$$l = \int_{\eta_0}^{\eta_1} \left(\frac{1}{c(\eta)} - \frac{1}{a}\right)^{-1} d\eta = \int_{\eta_1}^{\eta_0+\tau} \left(\frac{1}{c(\eta)} + \frac{1}{a}\right)^{-1} d\eta. \tag{3.3}$$

The periodic broken line exists if (3.3) has a real solution in η_0 and η_1.

As an illustration, let us take the previous law of parameter variation (2.5) and let us again examine the cases of "synchronized" and "asynchronous" waves. In the case of "synchronized" waves ($|1-\beta| < m$), we obtain from (3.3):

$$\operatorname{th}\alpha_1 = \frac{1}{\gamma_1} \frac{\operatorname{tg}\frac{\Omega\eta_1}{2} - \operatorname{tg}\frac{\Omega\eta_0}{2}}{1 - \frac{1}{\gamma_1^2}\operatorname{tg}\frac{\Omega\eta_1}{2}\operatorname{tg}\frac{\Omega\eta_0}{2}}, \quad \operatorname{tg}\beta_1 = -\frac{1}{\delta} \frac{\operatorname{tg}\frac{\Omega\eta_1}{2} - \operatorname{tg}\frac{\Omega\eta_0}{2}}{1 + \frac{1}{\delta^2}\operatorname{tg}\frac{\Omega\eta_1}{2}\operatorname{tg}\frac{\Omega\eta_0}{2}},$$

$$\alpha_1 = \frac{\pi}{2}\frac{\Omega}{\omega}\sqrt{m^2 - (1-\beta)^2}, \quad \beta_1 = \frac{\pi}{2}\frac{\Omega}{\omega}\sqrt{(1+\beta)^2 - m^2}, \tag{3.4}$$

$$\delta = \sqrt{\frac{1+\beta+m}{1+\beta-m}}, \quad \gamma_1 = \sqrt{\frac{1-\beta+m}{m-1+\beta}},$$

$\tau = 2\pi N/\Omega$, $N = 1, 2, \ldots$, $\omega = \pi c_0/l$ is the smallest eigenfrequency of the system with constant parameters. The solution (3.4) will be real if

$$(\delta^2 + \gamma_1^2)^2 \operatorname{th}^2\alpha_1 \operatorname{tg}^2\beta_1 + 4\gamma_1\delta(\delta\operatorname{tg}\beta_1 + \gamma_1\operatorname{th}\alpha_1)(\delta\operatorname{th}\alpha_1 - \gamma_1\operatorname{tg}\beta_1) \geqslant 0. \tag{3.5}$$

The form of the domain of existence of the periodic broken line is shown in this case in Fig. 5 for $\beta = 1$.

For the "asynchronous" waves ($|1-\beta| > m$), we have from (3.3):

$$\operatorname{tg}\alpha_2 = \frac{1}{\gamma_2} \frac{\operatorname{tg}\frac{\Omega\eta_1}{2} - \operatorname{tg}\frac{\Omega\eta_0}{2}}{1 + \frac{1}{\gamma_2^2}\operatorname{tg}\frac{\Omega\eta_0}{2}\operatorname{tg}\frac{\Omega\eta_1}{2}}, \tag{3.6}$$

$$\operatorname{tg}\beta_1 = -\frac{1}{\delta} \frac{\operatorname{tg}\frac{\Omega\eta_1}{2} - \operatorname{tg}\frac{\Omega\eta_0}{2}}{1 + \frac{1}{\delta^2}\operatorname{tg}\frac{\Omega\eta_0}{2}\operatorname{tg}\frac{\Omega\eta_1}{2}}, \quad \gamma_2 = \sqrt{\frac{1-\beta+m}{1-\beta-m}}, \quad \alpha_2 = \frac{\pi}{2}\frac{\Omega}{\omega}\sqrt{(1-\beta)^2 - m^2}.$$

The condition for the existence of real η_0, η_1 satisfying (3.6) has the form

$$(\delta^2 - \gamma_2^2)^2 \operatorname{tg}^2\alpha_2 \operatorname{tg}^2\beta_1 - 4\gamma_2\delta(\delta\operatorname{tg}\beta_1 + \gamma_2\operatorname{tg}\alpha_2)(\gamma_2\operatorname{tg}\beta_1 + \delta\operatorname{tg}\alpha_2) \geqslant 0. \tag{3.7}$$

The domain of existence of the periodic broken line in the plane of the parameters m, Ω/ω is shown in Fig. 6 for $\beta = 0.5$.

Both in the case of the "synchronized" and "asynchronous" waves (Figs. 5 and 6), the domains of existence of the periodic broken lines are sets of nonintersecting zones. Comparing the sizes of the domains defined by (3.5) and (3.7) shows that the zone width in the case of "synchronized" waves is approximately 5 times greater than in the case of the "asynchronous" waves.

2. Let us examine the condition for the families of the broken and the period lines to come together. From (3.2) we have

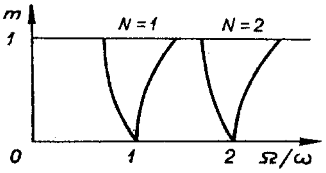

Fig. 5

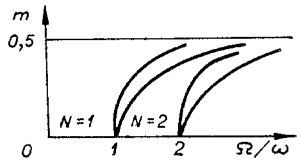

Fig. 6

$$\left|\frac{a+c(\eta_0)}{a+c(\eta_1)}\frac{a-c(\eta_1)}{a-c(\eta_0)}\right| < 1 \quad \text{or} \quad \mathrm{tg}^2\frac{\Omega\eta_0}{2} < \mathrm{tg}^2\frac{\Omega\eta_1}{2}. \tag{3.8}$$

It turns out that this condition is satisfied everywhere that (3.5) and (3.7) are, with the exception of the boundaries on which the equality $|\tan(\Omega\eta_0/2)| = |\tan(\Omega\eta_1/2)|$ holds. We therefore see that the instability domain is defined by (3.5) in the case of "synchronized" waves, and by (3.7) in the case of "asynchronous" waves.

Certain losses of mechanical energy always exist in real systems. Let us assume that they are lumped on the boundaries (Γ_a, Γ_b are the coefficients of wave reflection from the boundaries). Then we will have in place of (3.8)

$$\left|\frac{a+c(\eta_0)}{a+c(\eta_1)}\frac{a-c(\eta_1)}{a-c(\eta_0)}\right| < \Gamma_a\Gamma_b.$$

It follows from this inequality that the instability domain with the losses taken into account will be narrower. The threshold instability

$$m_t = \frac{1-\Gamma_a\Gamma_b}{1+\Gamma_a\Gamma_b}\frac{1-\beta}{2\beta}$$

is independent of the number of the zone N, which indicates the necessity of taking account, in principle, of higher instability zones.

The analysis performed showed that energy transport upward on the spectrum is characteristic for parametric instability of the second kind; i.e., vibrations of higher and higher frequency originate. As is known [13], the energy of high-frequency vibrations is absorbed most intensively by a material, being realized as heat and destruction of the microstructure. The vibration amplitude can here remain within technical specifications; however, it should be expected that the operational longevity of the elements in such regimes will be reduced sharply. In this sense, not only the study of the instability of the second kind is important but also the investigation of transients when this instability takes place for a brief time.

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